Precoding Design and Performance Bounds for Integer-Forcing MIMO Communication

Elad Domanovitz and Uri Erez

Tel Aviv University

February 8, 2018

Introduction

- The Single-User Multiple-Input Multiple-Output (MIMO) Gaussian channel has been the focus of extensive research

$$\boldsymbol{y}_c = \mathbf{H}_c \boldsymbol{x}_c + \boldsymbol{z}_c,$$

- $\mathbf{x}_c \in \mathbb{C}^{N_t}$ is the channel input vector
- $\mathbf{y}_c \in \mathbb{C}^{N_r}$ is the channel output vector
- $\mathbf{H}_c$ is an $N_r \times N_t$ complex channel matrix
  $\rightarrow$ Fixed over entire block length
- $\boldsymbol{z}_c \sim \mathcal{CSCN}(0, \mathbf{I})$
- Power constraint: $\mathbb{E}(\boldsymbol{x_c}^H \boldsymbol{x_c}) \leq N_t \cdot \mathrm{SNR}$

Introduction

- The MIMO Gaussian broadcast channel has also been widely studied for well over a decade now:

$$\boldsymbol{y}_c^i = \mathbf{H}_c^i \boldsymbol{x}_c + \boldsymbol{z}_c^i$$

- Private (only) Messages vs. Common (only) Messages
  - Capacity is known for both scenarios ✓
  - Practical schemes?
    - Private Message ✓ (DPC: Tomlinson...)
    - Common Message?

      $\Longrightarrow$ Single user: SVD or QR+SIC

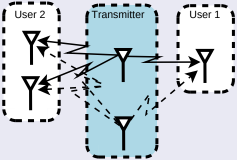
      $\Longrightarrow$ Two users: Solved using joint triangularization (Khina '12)

      $\Longrightarrow$ Moderate # of users: Extensions exist, not optimal (Khina '12)

      $\Longrightarrow$ Infinite # of users (knowing only WI-MI): Approximate joint triangularization is not very good $\Longrightarrow$ **Topic of this talk**

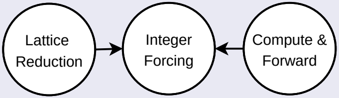
Objective

- Can we find a scheme that is:
  - Practical
    - Linear complexity in the block length
    - Uses off-the-shelf SISO codes
  - Has provable good performance guarantees
  - Universal: Is good for all channels with same WI-MI (compound channel setting), i.e., $\mathbf{H}_c \in \mathbb{H}(C_{\mathrm{WI}})$
- Universal $\Longrightarrow$ needs to deal with DoF mismatch

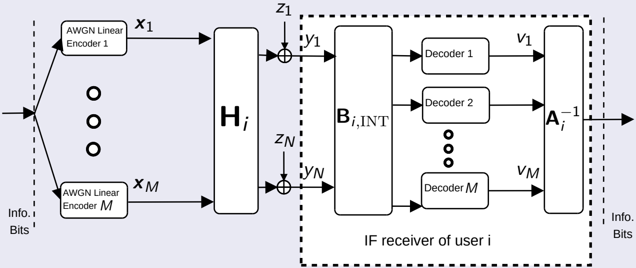


Candidate Scheme for OL-MIMO Broadcast: Integer Forcing

- Equalization scheme introduced by Zhan '10, et. al.



- Idea: Decode linear combination of messages $\Longrightarrow$ Invert



Integer-Forcing Equalization: Basic Idea

- Consider the (SU) channel

$$\mathbf{H} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- At high SNR linear receiver front-end *inverts* the channel (ZF) thus resulting in *noise amplification*

$$\mathbf{H}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \Longrightarrow \sigma_1^2 = 2,\ \sigma_2^2 = 5$$

- Can we avoid noise amplification?
- IF idea: If all streams are coded with **same** linear code $\Longrightarrow$

$$\text{Integer} \times \text{Codeword} + \text{Integer} \times \text{Codeword} = \text{Codeword}$$

- However, normal channels do not consist only of integers
- Integer Forcing (IF) equalization equalize the channel to he "nearest" integers-only matrix

Candidate Scheme for OL-MIMO Broadcast: Integer Forcing

- What is already known?
- Ordentlich '15, et. al.:
  - For single-user Open-Loop:
    - Rx side - Integer Forcing Equalization
    - Tx side - Space-Time Linear Precoding
  - ☺ A linear Non-Vanishing Determinant (NVD) precoder achieves the mutual information up to a constant gap for any channel
  - ☹ Guaranteed gap to capacity is quite large and does not provide satisfactory performance guarantees at moderate transmission rates

Bad Channels for IF/Linear Equalization

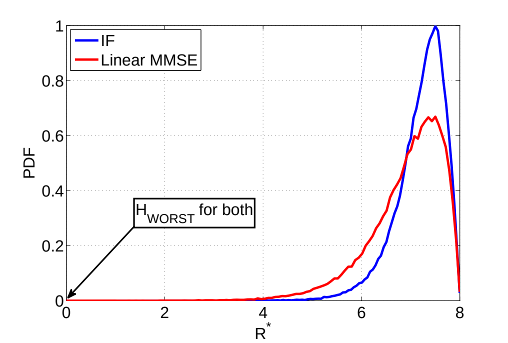


Figure: PDF of 2×2 Rayleigh channels normalized to WI=8 bits

- Worst channel $\mathbf{H}_{\text{worst}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$: one stream $\longrightarrow$ 

Combating Bad Channels via Random Precoding

- What can we do against nature?
- Apply random precoding

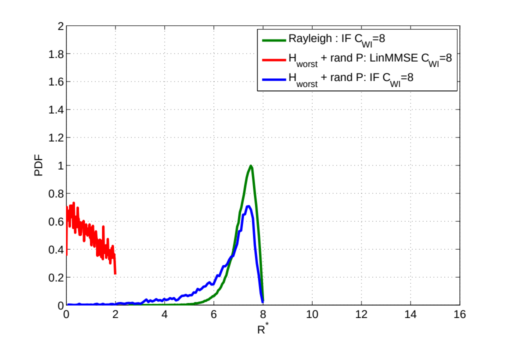


Figure: PDF of Random Unitary Precoding to $\mathbf{H}_{\text{worst}}$

Combating Bad Channels via Random Precoding

- What can we do against nature?
- Apply random precoding

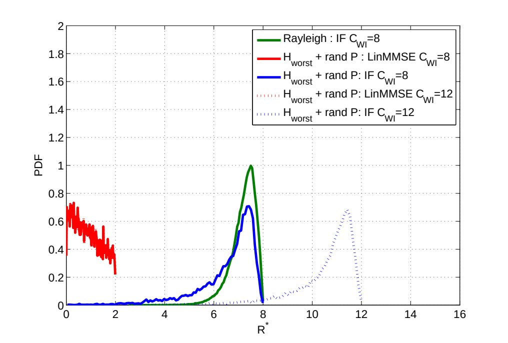


Figure: PDF of Random Unitary Precoding to $\mathbf{H}_{\text{worst}}$

Combating Bad Channels via Random Precoding

- What can we do against nature?
- Apply random precoding

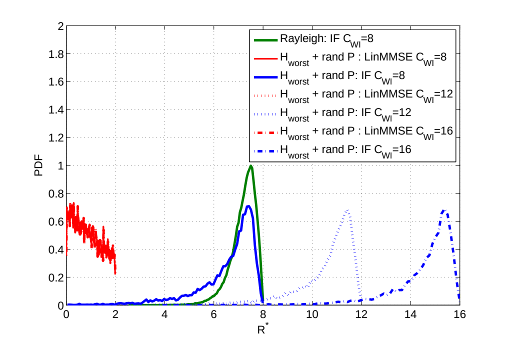


Figure: PDF of Random Unitary Precoding to $\mathbf{H}_{\text{worst}}$

Combating Bad Channels via Random Precoding

- What can we do against nature?
- Apply random precoding

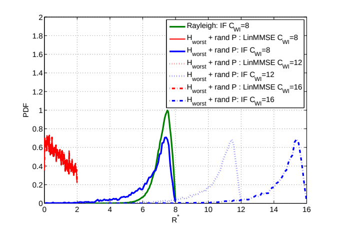


Figure: PDF of Random Unitary Precoding to $\mathbf{H}_{\text{worst}}$

- No precoding can salvage linear eq. when channel is singular
- IF copes well with channel being singular

Compound MIMO Channel Model

- $\mathbf{H}_c$ is part of the compound channel $\mathbb{H}(C_{\mathrm{WI}})$
- Mutual information of the compound channel is maximized by a Gaussian input with covariance matrix $\mathbf{Q}$:
  $C = \max_{\mathbf{Q}:\mathrm{Tr}\,\mathbf{Q} \leq N_t \mathrm{SNR}} \log\det\left(\mathbf{I}_{N_r \times N_r} + \mathbf{H}_c \mathbf{Q} \mathbf{H}_c^T\right)$
- We set $\mathrm{SNR} = 1 \implies \mathbf{H}_c = \mathbf{H}_c \sqrt{\mathrm{SNR}}$, taking $Q = I_{N_t \times N_t} \implies$
  $C_{\mathrm{WI}} = \log\det\left(\mathbf{I}_{N_r \times N_r} + \mathbf{H}_c \mathbf{H}_c^T\right)$
- Define:

$$\mathbb{H}(C_{\mathrm{WI}}) = \left\{\mathbf{H}_c \in \mathbb{C}^{N_r \times N_t} : \log\det\left(I + \mathbf{H}_c^T \mathbf{H}_c\right) = C_{\mathrm{WI}}\right\}$$

Compound MIMO Channel Model

- PDF figures$\Longrightarrow$ for **most** precoding matrices good performance, however there is a tail (outage)...
- In contrast to Rayleigh channel all channels in the compound class has same mutual information
  $\Longrightarrow$ Define (**scheme outage**) probability which is taken w.r.t. random precoding ensemble , not w.r.t. to channel statistics
- Instead of constant gap, our target is to bound

$$P_{\mathrm{out}}^{\mathrm{WC}}\left(C_{\mathrm{WI}}, R\right) = \sup_{\mathbf{H}_c \in \mathbb{H}(C_{\mathrm{WI}})} P\left(R_{\mathrm{IF}}\left(\mathbf{H}_c \cdot \mathbf{P}_c\right) < R\right)$$

- We assume $\mathbf{P}_c$ is drawn from CUE $\Longrightarrow$ channels with equal eigenvalues have equal outage probability

P-IF Performance

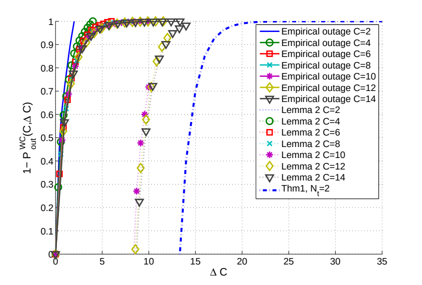


Figure: $N_r \times 2$ complex channel achievable probability

- Empirical = grid of singular values $\Longrightarrow$ large number of random unitary matrices $\Longrightarrow$ worst-case outage probability
- Lemma 2 = grid of singular values $\Longrightarrow$ summation over all $\mathbf{a} \in \mathbb{A}(R, N_t)$ $\Longrightarrow$ worst-case outage probability

P-IF Performance

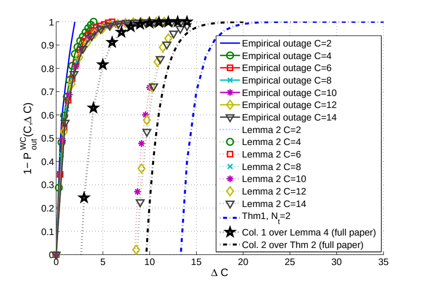


Figure: $N_r \times 2$ complex channel achievable probability

- Empirical = grid of singular values $\Longrightarrow$ large number of random unitary matrices $\Longrightarrow$ worst-case outage probability
- Lemma 2 = grid of singular values $\Longrightarrow$ summation over all $\mathbf{a} \in \mathbb{A}(R, N_t) \Longrightarrow$ worst-case outage probability

Universal Bound on Outage

Theorem

For any $N_r \times N_t$ complex channel with WI-MI C, and for $\mathbf{V}_c$ drawn from the CUE (which induces a real-valued precoding matrix $\mathbf{V}$), we have

$$P_{\mathrm{out}}^{\mathrm{WC}}(C, \Delta C) \leq c(N_t) 2^{-\Delta C},$$

where

$$c(N_t) = \left[\left(2 + \frac{\sqrt{2N_t}}{2}\right)^{2N_t} - \left(1 - \frac{\sqrt{2N_t}}{2}\right)^{2N_t} \right] N_t \alpha(N_t)^{N_t} \frac{\pi^{N_t}}{\Gamma(N_t+1)}$$

and

$$\alpha(N_t) = \frac{2N_t + 3}{4} \left(\frac{2}{\pi} \Gamma\left(2 + N_t\right)^{1/N_t} \right)^2.$$

Thus , $c(N_t)$ is a constant that depends only on N_t.

Application: Universal Bound for CL Achievable Rate

- A transmitter equipped with N_t transmit antennas wishes to send the same message to K users (each with $N_{r,i}$ antennas)
- Channel matrix of user i: $[\mathbf{H}_{c,i}]_{N_{r,i} \times N_t}$
- Set of channels: $\mathcal{H} = \{\mathbf{H}_{c,i}\}_{i=1}^{K}$
- Multicast capacity

$$C(\mathcal{H}) = \max_{\mathbf{Q}:\mathrm{Tr}(\mathbf{Q}) \leq N_t} \min_{\mathbf{H}_c \in \mathcal{H}} \log\det(\mathbf{I} + \mathbf{H}_c \mathbf{Q} \mathbf{H}_c^H)$$

- Absorbing the covariance matrix into the channel ($\hat{\mathbf{H}}_{c,i} = \mathbf{H}_{c,i}\mathbf{Q}^{1/2}$) we assume that optimal input covariance matrix is the identity matrix
- Thus

$$C(\mathcal{H}) = \min_i C_{\mathrm{WI}}(\hat{\mathbf{H}}_{c,i})$$

Application: Universal Bound for CL Achievable Rate

- WLOG assume equal WI-MI for all users $\implies \mathcal{H} \in \mathbb{H}(C_{\mathrm{WI}})$
- $A_i(R) =$ the event where a random precoding matrix $\widetilde{\mathbf{V}}_c$ drawn from a CUE achieves a desired target R for user i

$$A_i(R) = \left\{ \widetilde{\mathbf{V}}_c : R_{\mathrm{IF}}(\mathbf{H}_{c,i} \cdot \widetilde{\mathbf{V}}_c) \geq R \right\}$$

Union bound + probabilistic method:

- $P\left(\cap A_i(R)\right) \geq 1 - K P_{\mathrm{out}}^{\mathrm{WC}}(C(\mathcal{H}), R)$
- Guaranteed closed-loop rate

$$R_{\mathrm{WC-CL}}(\mathcal{H}) = \min_R \left(P_{\mathrm{out}}^{\mathrm{WC}}(C(\mathcal{H}), R) > \frac{1}{K} \right)$$

- Since

$$P\left(\cap A_i(R_{\mathrm{WC-CL}}(\mathcal{H}))\right) > 1 - K \cdot \frac{1}{K} = 0,$$

$\implies$ there exists a precoding matrix for which $R_{\mathrm{WC-CL}}(\mathcal{H})$ is achievable (via P-IF transmission)

Universal Bound for CL WC Rate

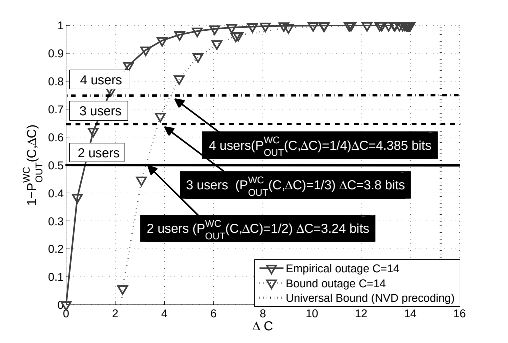


Figure: Guaranteed closed-loop performance for $N_r \times 2$ complex channels with 2,3,4 users

- Remark: Truth (as we believe...) is way better (Domanovitz '14 - outage with numerically optimized precoding matrix)